Problem 7. Let $\mathcal{H}$ be a Hilbert space.

(a) Show that the range of a compact operator $T \in K(\mathcal{H})$ is separable. (First, show the fact that every compact metric space is separable).

(b) Show that every compact operator $T \in K(\mathcal{H})$ can be approximated by operators of finite rank.

Hint: Let u_n be an orthonormal basis of $\operatorname{ran} T$ and P_n the orthogonal projection onto the linear hull $\mathcal{L}\{u_1, u_2, \dots, u_n\}$. Show that $\|A - P_n A\| \xrightarrow[n\to\infty]{} 0$.

(c) Let $\mathcal{J} \subseteq B(\mathcal{H})$ be a nontrivial closed ideal, i.e., a closed subspace with the property $AXB \in \mathcal{J}$ for $X \in \mathcal{J}$ and arbitrary operators $A, B \in B(\mathcal{H})$. Show that $\mathcal{J}$ contains all compact operators.

Problem 8. Let X, Y be Banach spaces and $T \in B(X, Y)$. Show that for every sequence $(x_n) \subseteq X$

$$x_n \xrightarrow{w} x \quad \Longrightarrow \quad Tx_n \xrightarrow{w} Tx.$$

If T is compact, then also

$$(\dagger) \qquad x_n \xrightarrow{w} x \quad \Longrightarrow \quad Tx_n \to Tx.$$

Problem 9. Let X be a separable Banach space. Show that there is a metric on the closed unit ball $\bar{B}_{X^*}(0,1)$ that produces the same open sets as the w*-Topology $\sigma(X^*, X)$.
Hint: Set $d(x^*, y^*) = \sum_{n=1}^{\infty} 2^{-n} |\langle x^* - y^*, x_n\rangle|$ for an appropriate sequence x_n.
Conclude from this that $(\bar{B}_X(0,1), \sigma(X, X^*))$ is also metrizable if X^* is separable.

Problem 10. Let X be a reflexive separable Banach space.

(a) Show that $(\bar{B}_X(0,1), \sigma(X, X^*))$ is metrizable and therefore a compact metric space.

(b) Let Y be another Banach space and $T \in B(X, Y)$ an operator with the property $(\dagger)$ from Problem 8. Show that T is compact.

Problem 11. Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$ be self-adjoint.

(a) Show that

$$\|T\| = \sup_{\|x\|\le 1} |\langle Tx, x\rangle|$$

Furthermore, let $x \in \mathcal{H}$ so that $\|Tx\| = \|T\|\,\|x\|$.

(b) Show that x is an eigenvector for T^2 with the eigenvalue $\lambda = \|T\|^2$.

(c) Show that T also has an eigenvector with eigenvalue $\lambda = \|T\|$ or $\lambda = -\|T\|$.